Self-similar Groups, Dynamical Systems and Algebras

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"Self-similar", or "fractal" objects are, in mathematics, those objects that are made, in a simple way, out of rescaled copies of themselves. One thinks, for instance, of a fern with a main stem and branches hanging off it; each of the branches is a small copy of the original fern.

This minicourse is an introduction to self-similarity in algebra. I will treat in some detail fundamental examples which have led to important developments in Group theory and Ring theory; and have had unexpected applications to Probability theory and Dynamical systems.

In the typical situations I will describe, an open problem is solved *using* self-similar algebra tools; but the statement of the answer does not require self-similar objects.

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The first two lectures will concentrate on groups and automata. I will describe a few classes of self-similar groups, in particular examples by Grigorchuk and Gupta & Sidki. I will explain how these groups answered major questions in group theory:

- Burnside's problem: does there exist a finitely generated infinite group in which every element has finite order?
- Does there exist a finitely generated group G that is isomorphic to its square $G \times G$? Or, at least, an infinite group G that contains $G \times G$ as a subgroup of finite index?

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- Milnor's problem: does there exist a finitely generated group G such that the number of group elements expressible as a word of length n in the generators has intermediate (subexponential, superpolynomial) growth rate?

For all three of these questions, the answer is "no" if one puts some restrictions on the group: it must be commutative, soluble, a group of matrices, ... Yet the answers are all "yes" in the world of groups, and the fundamental examples are self-similar.

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In the third lecture, I will describe V. Nekrashevych's connection between self-similar groups and dynamical systems. Given a topological dynamical system, more precisely a self-covering of a topological space, Nekrashevych associates with it a self-similar group which encodes fundamental features of the dynamical system.

In favourable cases (e.g. if the original dynamical system is metrically expanding, and the space has appropriate connectedness conditions) this group contains enough information that one can reconstruct the original dynamical system, up to homeomorphism.

I will explain in particular which self-similar groups are associated with quadratic polynomials, and detail the solution (found jointly with Nekrashevych) of the Hubbard's "twisted rabbit" problem.

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In the fourth lecture, I will describe how the self-similarity ideas introduced for groups can be applied to associative and Lie algebras. In particular, I will give examples of associative algebras answering a long-standing question in ring theory, and will describe the connections between self-similar groups and their related associative and Lie algebras.

In the spirit of Burnside's question, I will describe a Lie algebra that is ad-nil (repeated commutation with any given element always gives 0 after finitely many steps) but not nilpotent.

There are still many open problems and conjectures in this field, and I will attempt to indicate interesting future directions.