SPECIAL VALUES OF DIRICHLET SERIES ASSOCIATED TO POLYNOMIALS

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For f and g polynomials in p variables, we study special values obtained by analytic continuation of the Dirichlet series

$$\zeta(s; f, g) = \sum_{k_1=0}^{\infty} \cdots \sum_{k_p=0}^{\infty} g(k_1, \ldots, k_p) f(k_1, \ldots, k_p)^{-s}.$$

In particular, at $s = 0$, and under certain non-vanishing conditions on the polynomials involved, we prove the product rule

$$\text{degree}(fh) \cdot \zeta(0; fh, g) = \text{degree}(f) \cdot \zeta(0; f, g) + \text{degree}(h) \cdot \zeta(0; h, g). \qquad (*)$$

This generalizes formulas due to T. Shintani, K. Chen and M. Eie. In the proof we study zeta integrals of the form

$$Z(s; f, g) = \int_{x_1=0}^{\infty} \cdots \int_{x_p=0}^{\infty} g(x_1, \ldots, x_p) f(x_1, \ldots, x_p)^{-s} \, dx_p \cdots dx_1,$$

and show that $Z(0; f, g)$ can be computed in terms of a Laurent expansion of $g \cdot \log\bigl(f/f_{\text{top}}\bigr)$, where f_{top} is the top-degree homogeneous part of f. From this we deduce the product rule for zeta integrals

$$\text{degree}(fh) \cdot Z(0; fh, g) = \text{degree}(f) \cdot Z(0; f, g) + \text{degree}(h) \cdot Z(0; h, g).$$

Formula $(*)$ for Dirichlet series then follows from the Euler-Maclaurin formula. We also relate the values at negative integers of zeta integrals with those of Dirichlet series.

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